Synthesizing a Set of Cubes to Satisfy a Given Intersection Pattern*

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ABSTRACT

In two-level logic synthesis, the typical input specification is a set of minterms defining the on set and a set of minterms defining the don't care set of a Boolean function. The problem is to synthesize an optimal set of product terms, or cubes, that covers all the minterms in the on set and some of the minterms in the don't care set. In this paper, we consider a different specification: instead of the on set and the don't care set, we are given a set of numbers, each of which specifies the number of minterms covered by the intersection of one of the subsets of a set of λ cubes. We refer to the given set of numbers as an *intersection pattern*. The problem is to deterimine whether there exists a set of λ cubes to satisfy the given intersection pattern and, if it exists, to synthesize the set of cubes. We show a necessary and sufficient condition for the existence of λ cubes to satisfy a given intersection pattern. We also show that the synthesis problem can be reduced to the problem of finding a non-negative solution to a set of linear equalities and inequalities.

1. INTRODUCTION

Two-level logic synthesis is a well-developed and mature topic [1, 2]. The typical input specification for a two-level synthesis problem is the *on* set and the *don't care* set (or in some cases, the *off* set) of a Boolean function. The on set and the don't care set consist of minterms that define when the function evaluates to one and when its evaluation can be either zero or one, respectively. The problem is to synthesize an optimal set of product terms, or *cubes*, that covers all the minterms in the on set and some of the minterms in the don't care set.

In this work, we consider a related yet different problem pertaining to the synthesis of a set of cubes. A set of cubes, besides defining a Boolean function, also defines a set of numbers, each of which corresponds to the number of minterms covered by the intersection of one of the subsets of the set of cubes. For example, given a set of three cubes on four variables x_0, x_1, x_2, x_3, which are $c_0 = x_0 \wedge x_1$, $c_1 = x_2$, and $c_2 = x_1 \wedge x_3$, the numbers of minterms covered by c_0, c_1, c_2, $c_0 \wedge c_1$, $c_0 \wedge c_2$, $c_1 \wedge c_2$, and $c_0 \wedge c_1 \wedge c_2$ are 4, 8, 4, 2, 2, 2, and 1, respectively. We refer to this set of numbers as an *intersection pattern*.

Given a set of cubes, it is trivial to get its intersection pattern. However, it is nontrivial to answer the reverse problem: given a set of numbers that corresponds to an intersection pattern of λ cubes, how can one synthesize a set of λ cubes to satisfy the given intersection pattern, or prove that there is no solution to the given intersection pattern? We will call this the *λ-cube intersection problem*. It is what we intend to solve in this paper. We are interested in this problem since it is part of our broader effort to develop a synthesis methodology for probabilistic computation [3].

Definition 1
Define $V(f)$ to be the number of minterms contained in a Boolean function f. □

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Example 1
In a 3-cube intersection problem on 4 variables $x_0, \ldots, x_3$, if we are given the intersection pattern as

$$V(c_0) = 4, V(c_1) = 8, V(c_2) = 4,$$
$$V(c_0 \wedge c_1) = 2, V(c_0 \wedge c_2) = 2, V(c_1 \wedge c_2) = 2,$$
$$V(c_0 \wedge c_1 \wedge c_2) = 1,$$

we can synthesize a set of cubes $c_0 = x_0 \wedge x_1$, $c_1 = x_2$, and $c_2 = x_1 \wedge x_3$ to satisfy the intersection pattern. □

2. PRELIMINARIES

In this section, we will first introduce some basic definitions and then give a formal definition of the λ-cube intersection problem. Some of the basic definitions are adopted from [4].

The set of n *variables* of a Boolean function is denoted as $x_0, \ldots, x_{n-1}$. For a variable x, x and $\bar{x}$ are referred to as *literals*. A *Boolean product*, or *product*, is a conjunction of literals such that x and $\bar{x}$ do not appear simultaneously. A *minterm* is a Boolean product in which each of the n variables appear once, in either its complemented or uncomplemented form.

In the geometrical interpretation, a Boolean product is also known as a *cube*, denoted by c, and a minterm is also referred to as a *vertex* of the entire Boolean space. If cube c_2 takes the value one whenever cube c_1 equals one, we say that cube c_1 *implies* cube c_2 and write as $c_1 \subseteq c_2$. If cube c_1 implies cube c_2, then we have $V(c_1) \leq V(c_2)$. If $c_1 \wedge c_2 = 0$, we say that cube c_1 and c_2 are *disjoint*.

If a cube c contains k literals $(0 \leq k \leq n)$, then the number of vertices contained in the cube is $V(c) = 2^{n-k}$. Note that when a cube contains 0 literals, it is a special cube $c = 1$, which contains all vertices in the entire Boolean space. There is another special cube called *empty cube*, which is $c = 0$. The number of vertices contained in an empty cube is $V(c) = 0$. Thus, the number of vertices contained in a cube is in the set $S = \{s | s = 0 \text{ or } s = 2^k, k = 0, 1, \ldots, n\}$.

To make the representation compact, we use the following definitions.

Definition 2
Given a cube c and $\gamma \in \{0, 1\}$, define

$$c^\gamma = \begin{cases} 1, & \text{if } \gamma = 0 \\ c, & \text{if } \gamma = 1 \end{cases}$$

Given a set of λ cubes $c_0, \ldots, c_{\lambda-1}$ and an integer $\Gamma = \sum_{i=0}^{\lambda-1} \gamma_i 2^i$, where $\gamma_i \in \{0, 1\}$, define C^Γ to be the intersection of a subset of cubes c_i with $\gamma_i = 1$, i.e., $C^\Gamma = \bigwedge_{i=0}^{\lambda-1} c_i^{\gamma_i}$. □

Definition 3
Given an integer $\Gamma = \sum_{i=0}^{\lambda-1} \gamma_i 2^i$, where $\gamma_i \in \{0, 1\}$, define $B(\Gamma)$ to be the number of ones in the binary representation of Γ, i.e., $B(\Gamma) = \sum_{i=0}^{\lambda-1} \gamma_i$. □

With the above definition, we can more formally define the λ-cube intersection problem as follows:

Given $n > 0$, $\lambda > 0$, and $2^\lambda - 1$ numbers $v_1, v_2, \dots v_{2^\lambda-1} \in S = \{s | s = 0 \text{ or } s = 2^k, k = 0, 1, \dots, n\}$, determine whether there exists a set of λ cubes $c_0, \dots, c_{\lambda-1}$ on n variables $x_0, \dots, x_{n-1}$, such that for any $1 \le \Gamma \le 2^\lambda - 1$, $V(C^\Gamma) = v_\Gamma$.

We refer to the vector of numbers $(v_1, \dots, v_{2^\lambda-1})$ as *an intersection pattern* on λ cubes, or simply as an intersection pattern. If a set of λ cubes $c_0, \dots, c_{\lambda-1}$ satisfies the property that for any $1 \le \Gamma \le 2^\lambda - 1$, $V(C^\Gamma) = v_\Gamma$, then we say that the set of cubes satisfies the intersection pattern $(v_1, \dots, v_{2^\lambda-1})$.

For convenience, we represent a cube as a *cube-variable row vector* and a set of cubes as a *cube-variable matrix*. These are defined as follows.

Definition 4
Given a nonempty cube c on n variables $x_0, \dots, x_{n-1}$, we represent it by a cube-variable row vector *U of length n, whose elements are from the set $\{0, 1, *\}$. If the j-th ($0 \le j \le n-1$) element $U_j = 1$, then the literal x_j appears in the cube c; if $U_j = 0$, then the literal $\bar{x}_j$ appears in the cube c; if $U_j = *$, then the cube c does not depend on the variable x_j.*

Given a set of λ nonempty cubes $c_0, \dots, c_{\lambda-1}$ on n variables $x_0, \dots, x_{n-1}$, we represent them by a cube-variable matrix *D of size $\lambda \times n$, so that the i-th row of the matrix is the cube-variable row vector of c_i.* □

For example, a set of two cubes $c_0 = x_0 \wedge \bar{x}_1$ and $c_1 = \bar{x}_0 \wedge x_2$ is represented as a cube-variable matrix

$$\begin{bmatrix} 1 & 0 & * \\ 0 & * & 1 \end{bmatrix}$$

Given a cube-variable row vector, the following simple lemma suggests how to obtain the number of vertices covered by the corresponding cube.

Lemma 1
If the cube-variable row vector of a nonempty cube contains k $*$'s, then the cube covers 2^k number of vertices. □

Definition 5
For a value a in $\{0, 1, *\}$, the negation of a is defined as

$$\bar{a} = \begin{cases} 1, & \text{if } a = 0 \\ 0, & \text{if } a = 1 \\ *, & \text{if } a = * \end{cases}$$

The negation of a cube-variable matrix (column vector) is the element-wise negation of the matrix (column vector). □

In what follows, we will say that a cube-variable matrix satisfies the given intersection pattern if the corresponding set of cubes satisfies the intersection pattern. The following lemma is straightforward.

Lemma 2
Suppose that a cube-variable matrix D satisfies the intersection pattern $(v_1, \dots, v_{2^\lambda-1})$. Then D' satisfies the same intersection pattern if D' is obtained from D by column permutation or column negation. □

3. A SPECIAL CASE OF THE λ-CUBE INTERSECTION PROBLEM

Here we consider a specific case in which $v_{2^\lambda-1} > 0$. First, we have the following theorem, which gives a necessary condition for λ cubes to satisfy the given intersection pattern.

Theorem 1
If $v_{2^\lambda-1} > 0$ and there exist λ cubes to satisfy the λ-cube intersection problem, then for any $1 \le \Gamma \le 2^\lambda - 1$, v_Γ can be represented as $v_\Gamma = 2^{k_\Gamma}$, where $0 \le k_\Gamma \le n$ is an integer. □

PROOF. Based on Definition 2, for any $1 \le \Gamma \le 2^\lambda - 1$, $C^{2^\lambda-1} \subseteq C^\Gamma$. Therefore,

$$0 < v_{2^\lambda-1} = V(C^{2^\lambda-1}) \le V(C^\Gamma) = v_\Gamma.$$

Since for any $1 \le \Gamma \le 2^\lambda - 1$, $v_\Gamma \in S$ and $v_\Gamma > 0$, therefore, there exists an integer $0 \le k_\Gamma \le n$, such that $v_\Gamma = 2^{k_\Gamma}$. □

In what follows, we will assume that there exist λ cubes to satisfy the given intersection pattern. Then, there exist $2^\lambda - 1$ integers $k_1, \dots, k_{2^\lambda-1}$ such that for any $1 \le \Gamma \le 2^\lambda - 1$, $v_\Gamma = 2^{k_\Gamma}$. Further, notice that $V(C^0) = 2^n$. We let $v_0 = 2^n$ and $k_0 = n$.

Without loss of generality, we could assume that each entry of the cube-variable matrix is either 1 or $*$. Since $\bigwedge_{i=0}^{\lambda-1} c_i \neq 0$, then for each column of the matrix D, it does not simultaneously contain both a 0 and a 1. Otherwise, $\bigwedge_{i=0}^{\lambda-1} c_i = 0$. Therefore, each column of the matrix D contains either only 0's and $*$'s or only 1's and $*$'s. By Lemma 2, if we negate those columns of the matrix D that contain only 0's and $*$'s, then the new matrix D' obtained still satisfies the given intersection pattern. The matrix D' only contains 1's and $*$'s.

Definition 6
Given any $0 \le \Gamma \le 2^\lambda - 1$, suppose that $\Gamma = \sum_{i=0}^{\lambda-1} \gamma_i 2^i$, where $\gamma_i \in \{0, 1\}$. Define ψ_Γ to be a column vector of length λ with elements from the set $\{1, *\}$, such that the i-th element ($0 \le i \le \lambda - 1$) of it is

$$(\psi_\Gamma)_i = \begin{cases} 1, & \text{if } \gamma_i = 0 \\ *, & \text{if } \gamma_i = 1 \end{cases} \qquad \square$$

For example, if $\lambda = 3$, then $\psi_0 = (1, 1, 1)^T$ and $\psi_5 = (*, 1, *)^T$. Since each column of the cube-variable matrix only contains 1's and $*$'s, each column $D_{\cdot j}$ is in the set $\{\psi_0, \psi_1, \dots, \psi_{2^\lambda-1}\}$.

Definition 7
For any $0 \le \Gamma \le 2^\lambda - 1$, define J_Γ to be the set of indices of the columns in the matrix D of the form ψ_Γ, i.e., $J_\Gamma = \{j | D_{\cdot j} = \psi_\Gamma\}$. Define z_Γ to be the cardinality of the set J_Γ. □

Definition 8
Given two integers A and B, let their binary representation be $A = \sum_{i=0}^{k-1} a_i 2^i$ and $B = \sum_{i=0}^{k-1} b_i 2^i$, where $a_i, b_i \in \{0, 1\}$. We write $A \succeq B$ when for any $0 \le i \le k-1$, $a_i \ge b_i$. □

The following theorem gives relation between $\{z_0, \dots z_{2^\lambda-1}\}$ and $\{k_0, \dots, k_{2^\lambda-1}\}$.

Theorem 2
For any $0 \le L \le 2^\lambda - 1$, we have

$$k_L = \sum_{0 \le \Gamma \le 2^\lambda-1 : \Gamma \succeq L} z_\Gamma. \tag{1}$$

□

PROOF. Since the total number of columns in matrix D is n, we have $\sum_{\Gamma=0}^{2^\lambda-1} z_\Gamma = n = k_0$, or $\sum_{0 \le \Gamma \le 2^\lambda-1 : \Gamma \succeq 0} z_\Gamma = k_0$. Thus, Equation (1) holds for $L = 0$.

Now consider $1 \le L \le 2^\lambda - 1$. Then L can be represented as $L = \sum_{j=0}^{r-1} 2^{l_j}$, where $1 \le r \le \lambda$ and $0 \le l_0 < \dots < l_{r-1} \le \lambda - 1$. Then, C^L represents the intersection of the set of cubes $c_{l_0}, \dots, c_{l_{r-1}}$. The i-th entry in the cube-variable row vector of their intersection C^L is $*$ if and only if the column $D_{\cdot i}$ has $*$'s on the row $l_0, l_1, \dots, l_{r-1}$. Therefore, the number of $*$'s in the cube-variable row vector of their intersection C^L is the number of columns in D, whose entries on the row $l_0, l_1, \ldots, l_{r-1}$ are all $*$'s, or

$$\sum_{\substack{0 \le \Gamma \le 2^\lambda - 1: \\ (\psi_\Gamma)_{l_0} = \cdots = (\psi_\Gamma)_{l_{r-1}} = *}} z_\Gamma.$$

On the other hand,by Lemma 1, since $V(C^L) = 2^{k_L}$, the number of $*$'s in the cube-variable row vector of C^L is k_L. Therefore, we have

$$k_L = \sum_{\substack{0 \le \Gamma \le 2^\lambda - 1: \\ (\psi_\Gamma)_{l_0} = \cdots = (\psi_\Gamma)_{l_{r-1}} = *}} z_\Gamma = \sum_{\substack{0 \le \Gamma \le 2^\lambda - 1: \\ \gamma_{l_0} = \cdots = \gamma_{l_{r-1}} = 1}} z_\Gamma, \quad (2)$$

where $L = \sum_{j=0}^{r-1} 2^{l_j}$ and $\Gamma = \sum_{i=0}^{\lambda-1} \gamma_i 2^i$.

By Definition 8, we can rewrite Equation (2) as

$$k_L = \sum_{0 \le \Gamma \le 2^{\lambda-1} : \Gamma \succeq L} z_\Gamma.$$

□

Note that Equation (1) is a linear equation in $z_0, \ldots, z_{2^\lambda - 1}$ and holds for all $0 \le L \le 2^\lambda - 1$. Therefore, we can derive a system of 2^λ linear equations on unknowns $z_0, \ldots, z_{2^\lambda - 1}$:

$$\sum_{0 \le \Gamma \le 2^\lambda - 1 : \Gamma \succeq L} z_\Gamma = k_L, \text{ for } L = 0, 1, \ldots, 2^\lambda - 1. \quad (3)$$

We can represent the above system of linear equations in matrix form, as shown by the following theorem.

Theorem 3
Let vector $\vec{k} = (k_0, \ldots, k_{2^\lambda - 1})^T$ and vector $\vec{z} = (z_0, \ldots, z_{2^\lambda - 1})^T$. Then we can represent the system of 2^λ linear equations (3) in matrix form as

$$R_\lambda \vec{z} = \vec{k}, \quad (4)$$

where R_λ is a $2^\lambda \times 2^\lambda$ square matrix recursively defined as follows:

$$R_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, R_i = \begin{bmatrix} R_{i-1} & R_{i-1} \\ 0 & R_{i-1} \end{bmatrix}, \text{ for } i = 2, \ldots, \lambda. \quad \square$$

Due to space constraints, we omit the proof.

It is not hard to see that $\det(R_\lambda) = 1$. Therefore, R_λ is invertible. The following theorem shows what R_λ^{-1} is.

Theorem 4
R_λ^{-1} is recursively defined as follows:

$$R_1^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, R_i^{-1} = \begin{bmatrix} R_{i-1}^{-1} & -R_{i-1}^{-1} \\ 0 & R_{i-1}^{-1} \end{bmatrix}, \text{ for } i = 2, \ldots, \lambda. \quad \square$$

Therefore, given $k_0, k_1, \ldots, k_{2^\lambda - 1}$, we can get $z_0, z_1, \ldots, z_{2^\lambda - 1}$ as $\vec{z} = R_\lambda^{-1} \vec{k}$.

Since for any $0 \le \Gamma \le 2^\lambda - 1$, z_Γ is the cardinality of the set J_Γ, therefore, z_Γ must be a non-negative integer. By Theorem 4, R_λ^{-1} is an integer matrix. Therefore, $z_0, \ldots, z_{2^\lambda - 1}$ are always integers. Thus, a necessary condition for the existence of λ cubes to satisfy the given intersection pattern is that the vector $R_\lambda^{-1} \vec{k}$ has all entries non-negative. From Equation (4), we can see that the intersection pattern $(2^{k_1}, \ldots, 2^{k_{2^\lambda - 1}})$ only depends on $z_0, \ldots, z_{2^\lambda - 1}$. Therefore, as long as the vector $R_\lambda^{-1} \vec{k}$ has all entries non-negative, there exist λ cubes to satisfy the given intersection pattern. In fact, we can construct λ cubes with their cube-variable matrix as follows: for any column $0 \le j \le n-1$ of D, we can find a $0 \le \Gamma \le 2^\lambda - 1$ such that $\sum_{i=0}^{\Gamma-1} z_i \le j \le \sum_{i=0}^{\Gamma} z_i - 1$. Then, we let $D_{\cdot j} = \psi_\Gamma$. In summary, we have the following corollary.

Corollary 1
The necessary and sufficient condition for the existence of λ cubes to satisfy the given intersection pattern $(2^{k_1}, \ldots, 2^{k_{2^\lambda - 1}})$ is that the vector $R_\lambda^{-1} \vec{k}$ has all entries non-negative, where $\vec{k} = (n, k_1, \ldots, k_{2^\lambda - 1})^T$ and R_λ^{-1} is defined in Theorem 4. □

Example 2
Given $v_1 = 4$, $v_2 = 4$, and $v_3 = 1$, determine whether there exists a set of 2 cubes c_0 and c_1 on 4 variables to satisfy the intersection pattern (v_1, v_2, v_3).
Solution: *From the given conditions, we have $\vec{k} = (4, 2, 2, 0)^T$. Since*

$$R_2^{-1} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

then by Equation (4), we get $\vec{z} = (0, 2, 2, 0)^T$. Therefore, there are two ψ_1's and two ψ_2's in the cube-variable matrix of c_0 and c_1. One realization of the cube-variable matrix is

$$\begin{bmatrix} * & * & 1 & 1 \\ 1 & 1 & * & * \end{bmatrix}$$

and the corresponding cubes are $c_0 = x_2 \wedge x_3$ and $c_1 = x_0 \wedge x_1$.
□

4. GENERAL λ-CUBE INTERSECTION PROBLEM

In this section, we consider the more general situation where $v_{2^\lambda - 1} \ge 0$. Since we consider a set of nonempty cubes $c_0, \ldots, c_{\lambda-1}$, we assume that for any $0 \le i \le \lambda - 1$, $v_{2^i} = V(c_i) > 0$. Further, notice that $V(C^0) = 2^n$. We let $v_0 = 2^n$.

4.1 Necessary Conditions on the Positive v_Γ's

We first have the following theorem applicable for numbers $v_\Gamma > 0$.

Theorem 5
If there exist λ cubes $c_0, \ldots, c_{\lambda-1}$ to satisfy the intersection pattern, then for any $1 \le L \le 2^\lambda - 1$ such that $v_L > 0$, we have that for any $1 \le \Gamma \le 2^\lambda - 1$ such that $L \succeq \Gamma$, $v_\Gamma > 0$. □

PROOF. For any $1 \le \Gamma \le 2^\lambda - 1$ such that $L \succeq \Gamma$, it is not hard to see that $C^L \subseteq C^\Gamma$. Therefore,

$$0 < v_L = V(C^L) \le V(C^\Gamma) = v_\Gamma.$$

□

If a set of cubes is pairwise non-disjoint, then it has the following property.

Lemma 3
If a set of r cubes $c_{l_0}, \ldots, c_{l_{r-1}}$ $(3 \le r \le \lambda, 0 \le l_0 < \cdots < l_{r-1} \le \lambda - 1)$ is pairwise non-disjoint, i.e., for any $0 \le i < j \le r-1$, $c_{l_i} \wedge c_{l_j} \ne 0$, then their intersection $\bigwedge_{i=0}^{r-1} c_{l_i}$ is nonempty.
□

PROOF. By contraposition, suppose that $\bigwedge_{i=0}^{r-1} c_{l_i} = 0$. Consider the cube-variable matrix on these r cubes. Since their intersection is empty, there exists a column in the matrix that contains both a 0 and a 1. The cube corresponding to the 0 entry and the cube corresponding to the 1 entry are disjoint. This contradicts the assumption that the given set of cubes is pairwise non-disjoint. □

Alternatively, Lemma 3 can be stated on the numbers v_Γ. This gives a necessary condition for the existence of a set of cubes to satisfy the given intersection pattern.

Theorem 6
Suppose that there exist λ cubes $c_0, \ldots, c_{\lambda-1}$ to satisfy the given intersection pattern. If a set of r $(3 \le r \le \lambda)$ numbers $0 \le l_0 < \cdots < l_{r-1} \le \lambda - 1$ satisfies that for any $0 \le i < j \le r-1$, $v_{(2^{l_i}+2^{l_j})} > 0$, then for $L = \sum_{i=0}^{r-1} 2^{l_i}$, $v_L > 0$. □

For example, suppose that in a 4-cube intersection problem we are given $v_3 > 0$, $v_9 > 0$, and $v_{10} > 0$. If there exist 4 cubes to satisfy the given intersection pattern, then since $V(c_0 \wedge c_1) > 0$, $V(c_0 \wedge c_3) > 0$, and $V(c_1 \wedge c_3) > 0$, we must have $v_{11} = V(c_0 \wedge c_1 \wedge c_3) > 0$.

For the convenience, we first give the following definition.

Definition 9
Define

$$P = \{\Gamma | 0 \le \Gamma \le 2^\lambda - 1 \text{ and } v_\Gamma > 0\},$$
$$Z = \{\Gamma | 0 \le \Gamma \le 2^\lambda - 1 \text{ and } v_\Gamma = 0\}.$$

For any $0 \le i \le \lambda$, define

$$P_i = \{\Gamma | 0 \le \Gamma \le 2^\lambda - 1, B(\Gamma) = i, \text{ and } v_\Gamma > 0\},$$
$$Z_i = \{\Gamma | 0 \le \Gamma \le 2^\lambda - 1, B(\Gamma) = i, \text{ and } v_\Gamma = 0\}. \quad \square$$

From the definition of P and Z, we have the following obvious lemma, which gives a necessary condition on the existence of λ cubes to satisfy the given intersection pattern.

Lemma 4
If λ cubes $c_0, \ldots, c_{\lambda-1}$ satisfy the given intersection pattern, then for any $\Gamma \in P$, $C^\Gamma \neq 0$ and for any $\Gamma \in Z$, $C^\Gamma = 0$. □

However, by the following theorem, the above necessary condition could be reduced to the condition that for any $\Gamma \in P_2$, $C^\Gamma \neq 0$ and for any $\Gamma \in Z_2$, $C^\Gamma = 0$.

Theorem 7
Suppose that the given intersection pattern satisfies both Theorem 5 and 6:

1. *For any $1 \le L \le 2^\lambda - 1$, if $v_L > 0$, then for any $1 \le \Gamma \le 2^\lambda - 1$ such that $L \succeq \Gamma$, $v_\Gamma > 0$.*
2. *For any set of r $(3 \le r \le \lambda)$ numbers $0 \le l_0 < \cdots < l_{r-1} \le \lambda - 1$, if it satisfies that for any $0 \le i < j \le r-1$, $v_{(2^{l_i}+2^{l_j})} > 0$, then for the number $L = \sum_{i=0}^{r-1} 2^{l_i}$, $v_L > 0$.*

Then, a necessary and sufficient condition for a set of λ nonempty cubes to satisfy the condition that for any $\Gamma \in P$, $C^\Gamma \neq 0$ and for any $\Gamma \in Z$, $C^\Gamma = 0$ is that for any $\Gamma \in P_2$, $C^\Gamma \neq 0$ and for any $\Gamma \in Z_2$, $C^\Gamma = 0$. □

PROOF. The necessary part of the theorem is obvious, since the set P_2 is a subset of the set P and the set Z_2 is a subset of the set Z.

Now we prove the sufficient part. Suppose that a set of cubes satisfies that for any $\Gamma \in P_2$, $C^\Gamma \neq 0$ and for any $\Gamma \in Z_2$, $C^\Gamma = 0$.

It is not hard to see that the sets $P_0, \ldots, P_\lambda$ form a partition of the set P and that the sets $Z_0, \ldots, Z_\lambda$ form a partition of the set Z. Thus, we only need to prove that for all $0 \le k \le \lambda$, the set of cubes satisfies the condition that for any $\Gamma \in P_k$, $C^\Gamma \neq 0$ and for any $\Gamma \in Z_k$, $C^\Gamma = 0$.

We first consider the case that $k = 0$. By convention, $v_0 > 0$. Thus, $P_0 = \{0\}$ and $Z_0 = \phi$. Since $C^0 = 1$, thus we have that for any $\Gamma \in P_0$, $C^\Gamma \neq 0$. Since $Z_0 = \phi$, the statement that for any $\Gamma \in Z_0$, $C^\Gamma = 0$ also holds.

Now we consider the case that $k = 1$. Since we assume that for any $0 \le i \le \lambda - 1$, $v_{2^i} > 0$, therefore, $P_1 = \{2^i | i = 0, \ldots, \lambda - 1\}$ and $Z_1 = \phi$. Since $c_0, \ldots, c_{\lambda-1}$ are all nonempty, thus we have that for any $\Gamma \in P_1$, $C^\Gamma \neq 0$. Since $Z_1 = \phi$, the statement that for any $\Gamma \in Z_1$, $C^\Gamma = 0$ also holds.

When $k = 2$, the statement that the set of cubes satisfies that for any $\Gamma \in P_2$, $C^\Gamma \neq 0$ and for any $\Gamma \in Z_2$, $C^\Gamma = 0$ obviously holds.

Now we consider the case that $k \ge 3$. First, we consider any $L \in P_k$. Suppose that $L = \sum_{i=0}^{r-1} 2^{l_i}$, where $3 \le r \le \lambda$ and $0 \le l_0 < \cdots < l_{r-1} \le \lambda - 1$. Then, for any $0 \le i < j \le r-1$, $L \succeq (2^{l_i} + 2^{l_j})$. Therefore, based on the given condition, we have $v_{(2^{l_i}+2^{l_j})} > 0$. Since $B(2^{l_i} + 2^{l_j}) = 2$, thus $(2^{l_i} + 2^{l_j}) \in P_2$. By the assumption that for any $\Gamma \in P_2$, $C^\Gamma \neq 0$, we have that $C^{(2^{l_i}+2^{l_j})} = c_{l_i} \wedge c_{l_j} \neq 0$. Thus, the r cubes $c_{l_0}, \ldots, c_{l_{r-1}}$ are pairwise non-disjoint. By Lemma 3, then $C^L = \bigwedge_{i=0}^{r-1} c_{l_i} \neq 0$. Therefore, for any $L \in P_k$, $C^L \neq 0$.

Now we consider any $L \in Z_k$. Suppose that $L = \sum_{i=0}^{r-1} 2^{l_i}$, where $3 \le r \le \lambda$ and $0 \le l_0 < \cdots < l_{r-1} \le \lambda - 1$. We argue that there exist two numbers $0 \le u < v \le r-1$, such that $v_{(2^{l_u}+2^{l_v})} = 0$. Otherwise, for any $0 \le i < j \le r-1$, $v_{(2^{l_i}+2^{l_j})} > 0$. Then, based on the given conditions, we have $v_L > 0$. Therefore, it contradicts the assumption that $L \in Z_k$. Thus, there exist two numbers $0 \le u < v \le r-1$, such that $v_{(2^{l_u}+2^{l_v})} = 0$. Since $B(2^{l_u} + 2^{l_v}) = 2$, thus $(2^{l_u} + 2^{l_v}) \in Z_2$. By the assumption that for any $\Gamma \in Z_2$, $C^\Gamma = 0$, we have that $C^{(2^{l_u}+2^{l_v})} = c_{l_u} \wedge c_{l_v} = 0$. Thus, $C^L = \bigwedge_{i=0}^{r-1} c_{l_i} = 0$. Therefore, for any $L \in Z_k$, $C^L = 0$. □

For any $\Gamma \in P$, we assume $v_\Gamma = 2^{k_\Gamma}$, where $0 \le k_\Gamma \le n$ is an integer. Since $v_0 = 2^n$, we let $k_0 = n$. First, we give the following definition.

Definition 10
Given a cube-variable matrix D on λ cubes $c_0, \ldots, c_{\lambda-1}$, we define root cube-variable matrix *$t(D)$ of D as the cube-variable matrix formed by replacing the 0 entries in D with 1's and keeping the other entries in D unchanged. The set of cubes $c'_0, \ldots, c'_{\lambda-1}$ corresponding to the root matrix is called the set of* root cubes *to the original set of cubes.* □

For example, the root matrix of the cube-variable matrix

$$\begin{bmatrix} 1 & 0 & * \\ 0 & * & 1 \end{bmatrix} \quad \text{is} \quad \begin{bmatrix} 1 & 1 & * \\ 1 & * & 1 \end{bmatrix}.$$

The set of root cubes is $c'_0 = x_0 \wedge x_1$ and $c'_1 = x_0 \wedge x_2$.

Based on the definition of the set of root cubes, it is not hard to prove the following lemma.

Lemma 5
Suppose that the set of root cubes to the set of original cubes $c_0, \ldots, c_{\lambda-1}$ is $c'_0, \ldots, c'_{\lambda-1}$. Then, for any $\Gamma \in P$, we have $V(C'^\Gamma) = V(C^\Gamma)$. □

Since the root matrix $t(D)$ is a matrix containing only 1's and $*$'s, we can apply the definition of z_Γ in Definition 7 to $t(D)$. Then, based on the fact that for any $\Gamma \in P$, $V(C'^\Gamma) = V(C^\Gamma) = 2^{k_\Gamma}$, it is not hard to show that the following theorem characterizing the relation between z_Γ's and k_L's holds.

Theorem 8
If there exist λ cubes to satisfy the given intersection pattern, then for any $L \in P$,

$$\sum_{0 \le \Gamma \le 2^\lambda - 1 : \Gamma \succeq L} z_\Gamma = k_L. \quad \square$$

4.2 A Necessary and Sufficient Condition

In this section, we will show a necessary and sufficient condition for the existence of a set of cubes to satisfy the given intersection pattern. As a byproduct, the proof provides a way of synthesizing a set of cubes to satisfy the given intersection pattern. First, we define the *compatible column pattern set* for a number $\Gamma \in Z_2$.

Definition 11
Suppose that $\Gamma \in Z_2$ *and* $\Gamma = 2^i + 2^j$, *where* $0 \le i < j \le \lambda - 1$. *The compatible column pattern set for* Γ *is the set of column vectors* W *of length* λ *with entries from the set* $\{0, 1, *\}$, *such that*

1. $W_i = 0$ *and* $W_j = 1$ *or* $W_i = 1$ *and* $W_j = 0$,
2. *for any number* $L \in P_2$ *such that* $L = 2^k + 2^l$, *where* $0 \le k < l \le \lambda - 1$, *the situation that* $W_k = 0$ *and* $W_l = 1$ *or* $W_k = 1$ *and* $W_l = 0$ *does not happen.* □

It is not hard to see that if a cube-variable column vector is in the compatible column pattern set for a $\Gamma \in Z_2$, then the negation of that cube-variable column vector is also in that set. Therefore, we define the *representative compatible column pattern set* as follows.

Definition 12
The representative compatible column pattern set ρ_Γ *for* $\Gamma \in Z_2$ *is a subset of the compatible column pattern set for* Γ *such that the first non-*$*$ *entry of each element in the representative set is* 0. □

Example 3
Consider a 4-cube intersection problem with

$$P_2 = \{(0011)_2, (0101)_2, (1001)_2\},$$
$$Z_2 = \{(0110)_2, (1010)_2, (1100)_2\}.$$

The compatible column pattern set for $\Gamma = (0110)_2 \in Z_2$ *is*

$$\{(*010)^T, (*101)^T, (*011)^T, (*100)^T, (*01*)^T, (*10*)^T\}.$$

The representative compatible column pattern set for $\Gamma = (0110)_2$ *is* $\{(*010)^T, (*011)^T, (*01*)^T\}$. □

Definition 13
We define the set Y *as the union of the representative compatible column pattern sets* ρ_Γ *for all* $\Gamma \in Z_2$, *i.e.,* $Y = \bigcup_{\Gamma \in Z_2} \rho_\Gamma$. *We define the set* F *as the union of the set* Y *and the set of patterns contain only* 1*'s and* $*$*'s, i.e.,* $F = \bigcup_{i=0}^{2^\lambda - 1} \{\psi_i\} \cup Y$. □

Lemma 6
If there exists a cube-variable matrix D *to satisfy the given intersection pattern, then there exists another matrix* D' *which also satisfies the given intersection pattern and each column of which is in the set* F. □

PROOF. First, we argue that for any column of D which contains both a 0 and a 1 entry, the column is in the compatible column pattern set of a certain $\Gamma \in Z_2$. In fact, if a column r $(0 \le r \le n - 1)$ of D has the i-th entry being 0 and the j-th entry being 1, where $0 \le i, j \le \lambda - 1$ and $i \ne j$, then it is not hard to show that the column is in the compatible column pattern set for the number $(2^i + 2^j) \in Z_2$.

We can construct a D' from D as follows. For any column $0 \le r \le \lambda - 1$:

1. If $D_{\cdot r}$ contains only 1's and $*$'s, we let $D'_{\cdot r}$ be $D_{\cdot r}$. Then $D'_{\cdot r}$ is in the set $\bigcup_{i=0}^{2^\lambda - 1} \{\psi_i\}$.
2. If $D_{\cdot r}$ contains only 0's and $*$'s, we let $D'_{\cdot r}$ be the negation of the column $D_{\cdot r}$. Then $D'_{\cdot r}$ is in the set $\bigcup_{i=0}^{2^\lambda - 1} \{\psi_i\}$.
3. If $D_{\cdot r}$ contains both a 0 and a 1 and the first non-$*$ entry of $D_{\cdot r}$ is 0, we let $D'_{\cdot r}$ be $D_{\cdot r}$. Then, there exists a $\Gamma \in Z_2$ such that $D'_{\cdot r}$ is in the set ρ_Γ.
4. If $D_{\cdot r}$ contains both a 0 and a 1 and the first non-$*$ entry of $D_{\cdot r}$ is 1, we let $D'_{\cdot r}$ be the negation of the column $D_{\cdot r}$. Then, there exists a $\Gamma \in Z_2$ such that $D'_{\cdot r}$ is in the set ρ_Γ.

Then, by the above construction, each column of D' is in the set F. Further, D' is obtained from D by column negations. Thus, by Lemma 2, D' also satisfies the given intersection pattern. □

Based on Lemma 6, we only need to answer whether there exists a cube-variable matrix with columns from the set F to satisfy the given intersection pattern.

Lemma 7
If a cube-variable matrix D *with columns from the set* F *satisfies the given intersection pattern, then for any* $\Gamma \in Z_2$, *there exists a column in* D *which is in the set* ρ_Γ. □

PROOF. For any $\Gamma \in Z_2$, suppose that $\Gamma = 2^i + 2^j$, where $0 \le i < j \le \lambda - 1$. Since the cube-variable matrix satisfies the given intersection pattern, then based on Lemma 4, for the $\Gamma \in Z_2$, we must have $C^\Gamma = 0$ or $c_i \wedge c_j = 0$. Thus, there must exist a column r in D, such that $D_{ir} = 0$ and $D_{jr} = 1$ or $D_{ir} = 1$ and $D_{jr} = 0$. Now consider any $L \in P_2$. Suppose that $L = 2^k + 2^l$, where $0 \le k < l \le \lambda - 1$. Since the necessary condition for the cube-variable matrix to satisfy a given intersection pattern is that for the $L \in P_2$, $C^L \ne 0$, the situation that $D_{kr} = 0$ and $D_{lr} = 1$ or $D_{kr} = 1$ and $D_{lr} = 0$ cannot happen. Therefore, the column r of D is in the compatible column pattern set for Γ. Further, since all the columns of D are in the set F, then column r must be in the set ρ_Γ. □

By the similar definition of root cube-variable matrix, we define *root column vector* as follows.

Definition 14
Given a column vector W *with each element in the set* $\{0, 1, *\}$, *define its* root column vector $t(W)$ *as the column vector obtained from* W *by replacing the* 0 *entries in* W *with* 1*'s and keeping the other entries in* W *unchanged.* □

Definition 15
We define the set M *to be the set of numbers* $0 \le \Gamma \le 2^\lambda - 1$ *such that there exists an element in the set* Y, *whose root column vector is* ψ_Γ, *i.e.,*

$$M = \{\Gamma | 0 \le \Gamma \le 2^\lambda - 1, \text{ s.t. } \exists W \in Y \text{ s.t. } t(W) = \psi_\Gamma\}.$$

Define $\overline{M}$ *as* $\overline{M} = \{\Gamma | 0 \le \Gamma \le 2^\lambda - 1, \Gamma \notin M\}$.
For any $\Gamma \in M$, *we define the set* Y_Γ *to be the set of elements in the set* Y *such that their root column vectors are* ψ_Γ, *i.e.,* $Y_\Gamma = \{W | W \in Y \text{ and } t(W) = \psi_\Gamma\}$. □

Example 4
For the intersection pattern shown in Example 3, we have $Z_2 = \{6, 10, 12\}$ *and*

$$\rho_6 = \{(*010)^T, (*011)^T, (*01*)^T\},$$
$$\rho_{10} = \{(*001)^T, (*011)^T, (*0*1)^T\},$$
$$\rho_{12} = \{(*010)^T, (*001)^T, (**01)^T\}.$$

Thus,

$$Y = \{(*010)^T, (*001)^T, (*011)^T, (**01)^T, (*0*1)^T, (*01*)^T\},$$
$$M = \{1, 3, 5, 9\},$$

and $Y_1 = \{(*010)^T, (*001)^T, (*011)^T\}$, $Y_3 = \{(**01)^T\}$, $Y_5 = \{(*0*1)^T\}$, *and* $Y_9 = \{(*01*)^T\}$. □

Definition 16
For any $\Gamma \in M$, *we let the* $|Y_\Gamma|$ *elements in the set* Y_Γ *be* $\delta_{\Gamma,0}, \ldots, \delta_{\Gamma,|Y_\Gamma|-1}$. *For any* $0 \le i \le |Y_\Gamma| - 1$, *we define* $K_{\Gamma,i}$ *to be the set of indices of the columns in the matrix* D *of the form* $\delta_{\Gamma,i}$, *i.e.,* $K_{\Gamma,i} = \{k | D_{\cdot k} = \delta_{\Gamma,i}\}$. *We define* $w_{\Gamma,i}$ *to be the cardinality of the set* $K_{\Gamma,i}$. □

Theorem 9
Suppose that there exists a cube-variable matrix D *to satisfy the given intersection pattern, whose columns are from the set* F. *Then,*

we have that for any $\Gamma \in M$,

$$\sum_{i=0}^{|Y_\Gamma|-1} w_{\Gamma,i} \le z_\Gamma, \tag{5}$$

where z_Γ's are defined on the root matrix $t(D)$ according to Definition 7 and $w_{\Gamma,i}$'s are defined on the matrix D according to Definition 16. We also have that for any $L \in Z_2$,

$$\sum_{\substack{\Gamma\in M, 0\le i\le |Y_\Gamma|-1:\\ \delta_{\Gamma,i}\in\rho_L}} w_{\Gamma,i} \ge 1. \tag{6}$$

□

PROOF. Consider any $\Gamma \in M$. For any number $k \in \bigcup_{i=0}^{|Y_\Gamma|-1} K_{\Gamma,i}$, the column vector $D_{\cdot k}$ is in the set Y_Γ. Thus, the root column vector of $D_{\cdot k}$ is ψ_Γ. Thus, $k \in J_\Gamma$, where J_Γ is defined on the root matrix $t(D)$. Therefore, $\bigcup_{i=0}^{|Y_\Gamma|-1} K_{\Gamma,i} \subseteq J_\Gamma$. As a result, $\left|\bigcup_{i=0}^{|Y_\Gamma|-1} K_{\Gamma,i}\right| \le |J_\Gamma|$, or $\sum_{i=0}^{|Y_\Gamma|-1} w_{\Gamma,i} \le z_\Gamma$.

By Lemma 7, for any $L \in Z_2$, there exists a column in D which is in the set ρ_L. Suppose that column is of the form $\delta_{\Gamma^*,i^*} \in \rho_L$, where $\Gamma^* \in M$ and $0 \le i \le |Y_{\Gamma^*}| - 1$. Thus,

$$1 \le w_{\Gamma^*,i^*} \le \sum_{\substack{\Gamma\in M, 0\le i\le |Y_\Gamma|-1:\\ \delta_{\Gamma,i}\in\rho_L}} w_{\Gamma,i}.$$

□

Example 5
For the intersection pattern given in Example 3, based on the result shown in Example 4, we have

$$\delta_{1,0} = (*010)^T, \delta_{1,1} = (*001)^T, \delta_{1,2} = (*011)^T,$$
$$\delta_{3,0} = (**01)^T, \delta_{5,0} = (*0*1)^T, \delta_{9,0} = (*01*)^T.$$

The set of equations (5) for all $\Gamma \in M$ in this example is

$$\begin{cases} w_{\Gamma,0} \le z_\Gamma, \text{ for any } \Gamma \in \{3,5,9\} \\ w_{1,0} + w_{1,1} + w_{1,2} \le z_1 \end{cases}$$

The set of equations (6) for all $L \in Z_2$ in this example is

$$\begin{cases} w_{1,0} + w_{1,2} + w_{9,0} \ge 1 \\ w_{1,1} + w_{1,2} + w_{5,0} \ge 1 \\ w_{1,0} + w_{1,1} + w_{3,0} \ge 1 \end{cases} \quad \square$$

Finally, based on the necessary conditions for the existence of a cube-variable matrix to satisfy the given intersection pattern, shown in Theorem 5, 6, 8, and 9, we can derive a necessary and sufficient condition.

Theorem 10
There exists a cube-variable matrix D to satisfy the given intersection pattern $(v_1, \dots, v_{2^\lambda-1})$ if and only if

1. *for any $1 \le L \le 2^\lambda - 1$, if $v_L > 0$, then for any $1 \le \Gamma \le 2^\lambda - 1$ such that $L \succeq \Gamma$, $v_\Gamma > 0$,*

2. *for any set of r ($3 \le r \le \lambda$) numbers $0 \le l_0 < \cdots < l_{r-1} \le \lambda - 1$, if it satisfies that for any $0 \le i < j \le r-1$, $v_{(2^{l_i}+2^{l_j})} > 0$, then for the number $L = \sum_{i=0}^{r-1} 2^{l_i}$, $v_L > 0$, and*

3. *the system of equations on unknowns $\tilde{z}_\Gamma$ (for all $0 \le \Gamma \le 2^\lambda - 1$) and $\tilde{w}_{\Gamma,i}$ (for all $\Gamma \in M$ and $0 \le i \le |Y_\Gamma| - 1$)*

$$\begin{aligned} \sum_{0\le\Gamma\le 2^\lambda-1:\Gamma\succeq L} \tilde{z}_\Gamma &= k_L, \text{ for all } L \in P \\ \sum_{i=0}^{|Y_\Gamma|-1} \tilde{w}_{\Gamma,i} &\le \tilde{z}_\Gamma, \text{ for all } \Gamma \in M \\ \sum_{\substack{\Gamma\in M, 0\le i\le |Y_\Gamma|-1:\\ \delta_{\Gamma,i}\in\rho_L}} \tilde{w}_{\Gamma,i} &\ge 1, \text{ for all } L \in Z_2 \end{aligned} \tag{7}$$

has a non-negative integer solution. □

PROOF. "**only if**" part: Statement 1 in the theorem is due to Theorem 5 and Statement 2 in the theorem is due to Theorem 6.

Since D satisfies the given intersection pattern, then by Lemma 6, there exists another matrix D' which also satisfies the given intersection pattern and each column of which is in the set F. For any $0 \le \Gamma \le 2^\lambda - 1$, let $\tilde{z}_\Gamma = z_\Gamma$, where z_Γ's are defined on the root matrix $t(D')$ according to Definition 7. For any $\Gamma \in M$ and $0 \le i \le |Y_\Gamma| - 1$, let $\tilde{w}_{\Gamma,i} = w_{\Gamma,i}$, where $w_{\Gamma,i}$'s are defined on the matrix D' according to Definition 16. By Theorem 8 and 9, the set of numbers $\tilde{z}_\Gamma$ and $\tilde{w}_{\Gamma,i}$ satisfies the system of equations (7). Since $\tilde{z}_\Gamma$ is the cardinality of the set J_Γ and $\tilde{w}_{\Gamma,i}$ is the cardinality of the set $K_{\Gamma,i}$, therefore, $\tilde{z}_\Gamma$'s and $\tilde{w}_{\Gamma,i}$'s are all non-negative integers. Thus, the system of equations (7) has a non-negative solution.

"**if**" part: Let a non-negative solution to the system of equations (7) be $\tilde{z}_\Gamma = z_\Gamma$, for all $0 \le \Gamma \le 2^\lambda - 1$, and $\tilde{w}_{\Gamma,i} = w_{\Gamma,i}$, for all $\Gamma \in M$ and $0 \le i \le |Y_\Gamma| - 1$. Since for all $0 \le \Gamma \le 2^\lambda - 1$, $z_\Gamma \ge 0$, for all $\Gamma \in M$ and $0 \le i \le |Y_\Gamma| - 1$, $w_{\Gamma,i} \ge 0$, and for all $\Gamma \in M$, $\sum_{i=0}^{|Y_\Gamma|-1} w_{\Gamma,i} \le z_\Gamma$, then, we can construct a cube-variable matrix D so that

1. for all $\Gamma \in \overline{M}$, the matrix contains z_Γ columns of the form ψ_Γ,

2. for all $\Gamma \in M$, the matrix contains $z_\Gamma - \sum_{i=0}^{|Y_\Gamma|-1} w_{\Gamma,i}$ columns of the form ψ_Γ, and

3. for all $\Gamma \in M$ and all $0 \le i \le |Y_\Gamma| - 1$, the matrix contains $w_{\Gamma,i}$ columns of the form $\delta_{\Gamma,i}$.

All columns of the matrix D are in the set F. Next, we prove that the matrix D satisfies the given intersection pattern.

For any $L \in Z_2$, suppose $L = 2^i + 2^j$, where $0 \le i < j \le \lambda - 1$. Since $\sum_{\substack{\Gamma\in M, 0\le k\le |Y_\Gamma|-1:\\ \delta_{\Gamma,k}\in\rho_L}} w_{\Gamma,k} \ge 1$, there exists a $\Gamma^* \in M$ and a $0 \le k^* \le |Y_{\Gamma^*}| - 1$, such that $\delta_{\Gamma^*,k^*} \in \rho_L$ and $w_{\Gamma^*,k^*} \ge 1$. Therefore, the matrix D contains a column from the set ρ_L. Based on the definition of ρ_L, $C^L = c_i \wedge c_j = 0$. Thus, for any $L \in Z_2$, $C^L = 0$.

Now consider any $L \in P_2$. Suppose $L = 2^i + 2^j$, where $0 \le i < j \le \lambda - 1$. We argue that $C^L = c_i \wedge c_j \ne 0$. Otherwise, $c_i \wedge c_j = 0$. Therefore, there exists a column r in D, such $D_{ir} = 0$ and $D_{jr} = 1$ or $D_{ir} = 1$ and $D_{jr} = 0$. Since all the columns of D are in the set F, thus the column $D_{\cdot r}$ must be in the set Y. However, based on the definition of representative compatible column pattern set, each element W in the set Y satisfies that for the $L \in P_2$, the situation that $W_i = 0$ and $W_j = 1$ or $W_i = 1$ and $W_j = 0$ does not happen. Therefore, the column $D_{\cdot r}$ does not belong to the set Y. We get a contradiction. Thus, for any $L \in P_2$, we have $C^L \ne 0$.

Since the given intersection pattern satisfies the conditions of Theorem 7, then, based on Theorem 7, we have that for any $\Gamma \in Z$, $C^\Gamma = 0$ and for any $\Gamma \in P$, $C^\Gamma \ne 0$. Thus, for all these $\Gamma \in Z$, $V(C^\Gamma) = v_\Gamma = 0$.

Now consider any $L \in P$. When $L = 0$, it is not hard to see that the total number of columns in D is n.

For any $L \in P$ and $L > 0$, L can be represented as $L = \sum_{j=0}^{r-1} 2^{l_j}$, where $1 \le r \le \lambda$ and $0 \le l_0 < \cdots < l_{r-1} \le \lambda - 1$.

Since $C^L \neq 0$, the number of $*$'s in the cube-variable row vector C^L is the number of columns in D, whose entries on the row $l_0, l_1, \ldots, l_{r-1}$ are all $*$'s. Note that for any $0 \leq \Gamma \leq 2^\lambda - 1$, the column pattern ψ_Γ has all entries on the row $l_0, l_1, \ldots, l_{r-1}$ being $*$'s if and only if $\Gamma \succeq L$. Since the root column vector of $\delta_{\Gamma,i}$ is ψ_Γ, thus for any $\Gamma \in M$ and any $0 \leq i \leq |Y_\Gamma| - 1$, the column pattern $\delta_{\Gamma,i}$ has all entries on the row $l_0, l_1, \ldots, l_{r-1}$ being $*$'s if and only if $\Gamma \succeq L$. Therefore, the number of columns in D, whose entries on the row $l_0, l_1, \ldots, l_{r-1}$ are all $*$'s, is

$$\sum_{\substack{\Gamma \in \overline{M}: \\ \Gamma \succeq L}} z_\Gamma + \sum_{\substack{\Gamma \in M: \\ \Gamma \succeq L}} \left(z_\Gamma - \sum_{i=0}^{|Y_\Gamma|-1} w_{\Gamma,i} \right) + \sum_{\substack{\Gamma \in M: \\ \Gamma \succeq L}} \sum_{i=0}^{|Y_\Gamma|-1} w_{\Gamma,i}$$
$$= \sum_{0 \leq \Gamma \leq 2^\lambda - 1 : \Gamma \succeq L} z_\Gamma = k_L.$$

Therefore, the number of $*$'s in the row vector C^L is k_L. Since $C^L \neq 0$, by Lemma 1, $V(C^L) = 2^{k_L}$. Thus, for any $L \in P$ and $L > 0$, $V(C^L) = 2^{k_L} = v_L$.

In summary, the matrix D has n columns and for any $1 \leq \Gamma \leq 2^\lambda - 1$, $V(C^\Gamma) = v_\Gamma$. Thus, the matrix D satisfies the given intersection pattern. □

Comment: The above proof provides a way of synthesizing a cube-variable matrix to satisfy the given intersection pattern when the three conditions are all satisfied.

Example 6
In a 3-cube intersection problem on 4 variables $x_0, \ldots, x_3$, suppose that the intersection pattern is given as

$$v_1 = 4, v_2 = 4, v_3 = 0, v_4 = 4, v_5 = 1, v_6 = 2, v_7 = 0.$$

First, it is not hard to check that both Statement 1 and Statement 2 in Theorem 10 hold for the given pattern.

By convention, $v_0 = 2^4 = 16$. Therefore, we have

$$P = \{0, 1, 2, 4, 5, 6\}, \qquad Z = \{3, 7\},$$
$$k_0 = 4, k_1 = 2, k_2 = 2, k_4 = 2, k_5 = 0, k_6 = 1.$$

For the given intersection pattern, we have $Z_2 = \{3\}$ and $\rho_3 = \{(01*)^T\}$.

Thus, $Y = \{(01*)^T\}, M = \{4\}$ and $Y_4 = \{(01*)^T\}$. Thus, $\delta_{4,0} = (01*)^T$.

The system of equations (7) in this example is

$$\begin{aligned} &\tilde{z}_0 + \tilde{z}_1 + \tilde{z}_2 + \tilde{z}_3 + \tilde{z}_3 + \tilde{z}_4 + \tilde{z}_6 + \tilde{z}_7 = 4, \\ &\tilde{z}_1 + \tilde{z}_3 + \tilde{z}_5 + \tilde{z}_7 = 2, \quad \tilde{z}_2 + \tilde{z}_3 + \tilde{z}_6 + \tilde{z}_7 = 2, \\ &\tilde{z}_4 + \tilde{z}_5 + \tilde{z}_6 + \tilde{z}_7 = 2, \quad \tilde{z}_5 + \tilde{z}_7 = 0, \quad \tilde{z}_6 + \tilde{z}_7 = 1. \\ &\tilde{w}_{4,0} \leq \tilde{z}_4, \quad \tilde{w}_{4,0} \geq 1 \end{aligned} \tag{8}$$

The above system of equations (8) has a non-negative solution

$$\tilde{z}_1 = \tilde{z}_3 = \tilde{z}_4 = \tilde{z}_6 = 1, \tilde{z}_0 = \tilde{z}_2 = \tilde{z}_5 = \tilde{z}_7 = 0, \tilde{w}_{4,0} = 1.$$

Thus, Statement 3 in Theorem 10 also holds. Therefore, there exists a cube-variable matrix to satisfy the given intersection pattern. Based on the proof of Theorem 10, we can synthesize a cube-variable matrix that satisfies the given intersection pattern based on the above non-negative solution as

$$\begin{bmatrix} * & * & 0 & 1 \\ 1 & * & 1 & * \\ 1 & 1 & * & * \end{bmatrix}$$

and the corresponding cubes are $c_0 = \bar{x}_2 \wedge x_3$, $c_1 = x_0 \wedge x_2$, and $c_2 = x_0 \wedge x_1$. It is not hard to verify that the set of cubes c_0, c_1, c_2 satisfies the given intersection pattern. □

5. IMPLEMENTATION

In this section, we will discuss the implementation of the procedure to solve the λ-cube intersection problem, based on the theories in Section 4.

5.1 Checking Statement 1 in Theorem 10

For Statement 1 in Theorem 10, we can represent it in an alternative way, which is shown by the following theorem.

Theorem 11
The following two statements are equivalent:

1. *The intersection pattern $(v_1, \ldots, v_{2^\lambda - 1})$ satisfies that for any $1 \leq L \leq 2^\lambda - 1$, if $v_L > 0$, then for any $1 \leq \Gamma \leq 2^\lambda - 1$ such that $L \succeq \Gamma$, $v_\Gamma > 0$.*
2. *The intersection pattern $(v_1, \ldots, v_{2^\lambda - 1})$ satisfies that for any $2 \leq k \leq \lambda$ and any $L \in P_k$, if $1 \leq \Gamma \leq 2^\lambda - 1$ satisfies that $B(\Gamma) = k - 1$ and $L \succeq \Gamma$, then $v_\Gamma > 0$.* □

Based on Theorem 11, in order to check whether Statement 1 in Theorem 10 holds, we only need to check whether Statement 2 in Theorem 11 holds. Thus, whether Statement 1 in Theorem 10 holds can be checked by the procedure shown in Algorithm 1.

Algorithm 1 CheckRuleOne(λ, v): the procedure to check whether Statement 1 in Theorem 10 holds. It returns 1 if the statement holds; otherwise, it returns 0.

{Given an integer $\lambda \geq 1$ and a non-negative integer array $v = (v_1, \ldots, v_{2^\lambda - 1})$.}
for $i \Leftarrow 1$ to λ **do**
  $P_i \Leftarrow \{\Gamma | 0 \leq \Gamma \leq 2^\lambda - 1, B(\Gamma) = i,$ and $v_\Gamma > 0\}$;
for $i \Leftarrow 2$ to λ **do**
  for all $L \in P_i$ **do**
    for all $1 \leq \Gamma \leq 2^\lambda - 1$ s.t. $L \succeq \Gamma$ and $B(\Gamma) = i - 1$ **do**
      if $v_\Gamma = 0$ **then return** 0;
return 1;

5.2 Checking Statement 2 in Theorem 10

Whether Statement 2 in Theorem 10 holds can be checked by representing the given intersection pattern by an undirected graph and listing all maximal cliques of the undirected graph.

For a given intersection pattern on λ cubes, we can construct an undirected graph $G(N, E)$ from that pattern, where N is a set of λ nodes $n_0, \ldots, n_{\lambda-1}$ and E is a set of edges. There is an edge between the node n_i and n_j $(0 \leq i < j \leq \lambda - 1)$ if and only if the number $(2^i + 2^j)$ is in the set P_2.

For example, we can represent the intersection pattern shown in Example 3 by the undirected graph shown in Figure 1.

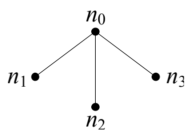

Figure 1: An undirected graph constructed from the intersection pattern of Example 3.

In graph theory, a *clique* in an undirected graph $G(N, E)$ is defined as a subset Q of the node set N, such that for every two nodes in Q, there exists an edge connecting the two. A *maximal clique* is a clique that cannot be extended by including one more adjacent node.

For an intersection pattern, if a set of r $(3 \leq r \leq \lambda)$ numbers $0 \leq l_0 < \cdots < l_{r-1} \leq \lambda - 1$ satisfies that for any $0 \leq i < j \leq r - 1$, $v_{(2^{l_i} + 2^{l_j})} > 0$, then, the set of nodes $n_{l_0}, \ldots, n_{l_{r-1}}$ forms a clique of the undirected graph constructed from the intersection pattern. Thus, Statement 2 in Theorem 10 can be stated in another way as: For any clique $Q = \{n_{l_0}, \ldots, n_{l_{r-1}}\}$ of size r in the undirected graph constructed from the intersection pattern, where $3 \leq r \leq \lambda$ and $0 \leq l_0 < \cdots < l_{r-1} \leq \lambda - 1$, the number $v_L > 0$, where $L = \sum_{i=0}^{r-1} 2^{l_i}$. In fact, it is not hard to see that if Statement 1 in Theorem 10 holds, then Statement 2 in Theorem 10 holds if and only if for any *maximal* clique $Q^* = \{n_{d_0}, \ldots, n_{d_{t-1}}\}$ of size t in the undirected graph constructed from the intersection pattern,

Algorithm 2 CheckRuleTwo(λ, v): the procedure to check whether Statement 2 in Theorem 10 holds under the assumption that Statement 1 in Theorem 10 holds. It returns 1 if the statement holds; otherwise, it returns 0.

{Given an integer $\lambda \geq 1$ and a non-negative integer array $v = (v_1, \dots, v_{2^\lambda - 1})$.}
$N \Leftarrow \{n_0, \dots, n_{\lambda-1}\}$; $E \Leftarrow \phi$;
for $i \Leftarrow 0$ to $\lambda - 1$ **do**
  for $j \Leftarrow i + 1$ to $\lambda - 1$ **do**
    if $v_{(2^i + 2^j)} > 0$ **then** $E \Leftarrow E \cup \{e(n_i, n_j)\}$;
for all maximal clique Q in the graph $G(N, E)$ **do**
  $L \Leftarrow \sum_{i: n_i \in Q} 2^i$;
  if $v_L = 0$ **then return** 0;
return 1;

where $3 \leq t \leq \lambda$ and $0 \leq d_0 < \cdots < d_{t-1} \leq \lambda - 1$, the number $v_{L^*} > 0$, where $L^* = \sum_{i=0}^{t-1} 2^{d_i}$. Therefore, whether Statement 2 in Theorem 10 holds can be answered by checking whether all v_L's corresponding to all maximal cliques in the undirected graph $G(N, E)$ are greater than zero. The problem of listing all maximal cliques in an undirected graph is a classical problem in graph theory and can be solved, for example, by the Born-Kerbosch algorithm [5].

Assuming that Statement 1 in Theorem 10 holds, then whether Statement 2 in Theorem 10 holds can be checked by the procedure shown in Algorithm 2.

5.3 Checking Statement 3 in Theorem 10

The following theorem shows that to check whether the system of equations (7) has a non-negative solution, we only need to check whether an alternative system of equations with fewer unknowns has a non-negative solution.

Theorem 12
The system of equations (7) *has a non-negative integer solution if and only if the system of equations on unknowns* $\hat{z}_\Gamma$ (*for all* $\Gamma \in \overline{M}$) *and* $\hat{w}_{\Gamma,i}$ (*for all* $\Gamma \in M$ *and* $0 \leq i \leq |Y_\Gamma| - 1$)

$$\begin{gathered} \sum_{\Gamma \in \overline{M}, \Gamma \succeq L} \hat{z}_\Gamma + \sum_{\Gamma \in M, \Gamma \succeq L} \sum_{i=0}^{|Y_\Gamma|-1} \hat{w}_{\Gamma,i} = k_L, \text{ for all } L \in P \\ \sum_{\substack{\Gamma \in M, 0 \leq i \leq |Y_\Gamma|-1: \\ \delta_{\Gamma,i} \in \rho_L}} \hat{w}_{\Gamma,i} \geq 1, \text{ for all } L \in Z_2 \end{gathered} \tag{9}$$

has a non-negative integer solution. □

Due to space constraints, we omit the proof here.

Based on Theorem 12, to check whether Statement 3 in Theorem 10 holds, we only need to check whether the system of equations (9) has a non-negative solution. Note that the system of equations (9) has $|M|$ fewer unknowns and $|M|$ fewer inequalities than the original system of equations (7). Thus, a certain amount of computation will be saved.

5.4 The Procedure to Solve the λ-Cube Intersection Problem

Based on the above discussion, we give the procedure to solve the λ-cube intersection problem in Algorithm 3. In the procedure, the function CheckRuleOne(λ, v) and the function CheckRuleTwo(λ, v) are shown in Algorithm 1 and 2, respectively. The function RCCPS(Γ, λ, P_2) returns the representative compatible column pattern set for a $\Gamma \in Z_2$. The function

SetEqn$(P, Z_2, M, \overline{M}, \{k_L | L \in P\}, \{\rho_L | L \in Z_2\}, \{Y_L | L \in M\})$

returns the matrices A_{ze}, A_{we}, A_w and the column vectors b_e and b in the matrix representation of the system of equations (9), which is

$$\begin{cases} A_{ze}\vec{z} + A_{we}\vec{w} = b_e, \\ A_w \vec{w} \geq b, \end{cases} \tag{10}$$

where $\vec{z}$ is a column vector of unknowns $\hat{z}_\Gamma$, for all $\Gamma \in \overline{M}$, and $\vec{w}$ is a column vector of unknowns $\hat{w}_{\Gamma,i}$, for all $\Gamma \in M$ and $0 \leq i \leq |Y_\Gamma| - 1$. The function NonNegSln$(A_{ze}, A_{we}, b_e, A_w, b)$ finds a non-negative integer solution to the system of equations (10). If the system of equations (10) has a non-negative integer solution, then the function returns one; otherwise, it returns ϕ. Given a non-negative solution $(\vec{z}, \vec{w})$ to the system of equations (10), the function SynCubes$(\vec{z}, \vec{w}, \lambda)$ synthesizes a set of λ cubes from that solution based on the proof of Theorem 10.

Algorithm 3 CubePattern(n, λ, v): the procedure to check whether there exists a set of λ cubes on n variables to satisfy the given intersection pattern $v = (v_1, \dots, v_{2^\lambda - 1})$. If the answer is yes, the procedure returns a set of cubes that satisfies the intersection pattern; otherwise, it returns ϕ.

{Given integers $n \geq 1$, $\lambda \geq 1$, and a non-negative integer array $v = (v_1, \dots, v_{2^\lambda - 1})$, where $v_\Gamma \in \{0, 2^0, 2^1, \dots 2^n\}$.}
$P \Leftarrow \phi$; $Z \Leftarrow \phi$;
for $i \Leftarrow 1$ to $2^\lambda - 1$ **do**
  if $v_\Gamma > 0$ **then** $P \Leftarrow P \cup \{\Gamma\}$; $k_\Gamma \Leftarrow \log_2 v_\Gamma$;
  else $Z \Leftarrow Z \cup \{\Gamma\}$;
if CheckRuleOne$(\lambda, v) = 0$ **then return** ϕ;
if CheckRuleTwo$(\lambda, v) = 0$ **then return** ϕ;
$P_2 \Leftarrow \{\Gamma | 0 \leq \Gamma \leq 2^\lambda - 1, B(\Gamma) = 2, \text{ and } v_\Gamma > 0\}$;
$Z_2 \Leftarrow \{\Gamma | 0 \leq \Gamma \leq 2^\lambda - 1, B(\Gamma) = 2, \text{ and } v_\Gamma = 0\}$;
for all $\Gamma \in Z_2$ **do** $\rho_\Gamma =$ RCCPS(Γ, λ, P_2);
$Y \Leftarrow \bigcup_{\Gamma \in Z_2} \rho_\Gamma$;
$M \Leftarrow \{\Gamma | 0 \leq \Gamma \leq 2^\lambda - 1, \text{ s.t. } \exists W \in Y \text{ s.t. } t(W) = \psi_\Gamma\}$;
$\overline{M} \Leftarrow \{\Gamma | 0 \leq \Gamma \leq 2^\lambda - 1, \Gamma \notin M\}$;
for all $\Gamma \in M$ **do** $Y_\Gamma \Leftarrow \{W | W \in Y \text{ and } t(W) = \psi_\Gamma\}$;
$(A_{ze}, A_{we}, b_e, A_w, b) \Leftarrow$ SetEqn$(P, Z_2, M, \overline{M},$
  $\{k_L | L \in P\}, \{\rho_L | L \in Z_2\}, \{Y_L | L \in M\})$;
$(\vec{z}, \vec{w}) \Leftarrow$ NonNegSln$(A_{ze}, A_{we}, b_e, A_w, b)$;
if $(\vec{z}, \vec{w}) = \phi$ **then return** ϕ;
return SynCubes$(\vec{z}, \vec{w}, \lambda)$;

6. CONCLUSION

In this paper, we introduced a new problem, the λ-cube intersection problem: Given a set of numbers corresponding to an intersection pattern of a set of λ cubes, we are asked to synthesize a set of cubes to satisfy the given intersection pattern, or prove that there is no such a solution to the problem. We provide a rigorous mathematic treatment to this problem and derive a necessary and sufficient condition for the existence of a set of cubes to satisfy the given intersection pattern. The problem reduces to two subproblems of listing all maximal cliques in an undirected graph and checking whether a set of linear equalities and inequalities has a non-negative integer solution. As a future work, we will use the algorithm presented to solve the λ-cube intersection problem as a subroutine to solve our more broader problem of synthesizing combinational logic for probabilistic computation.

7. REFERENCES

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